# The crossover from the Macroscopic Fluctuation Theory to the Kardar-Parisi-Zhang equation controls the large deviations of diffusive systems

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Quantinuum & ENS

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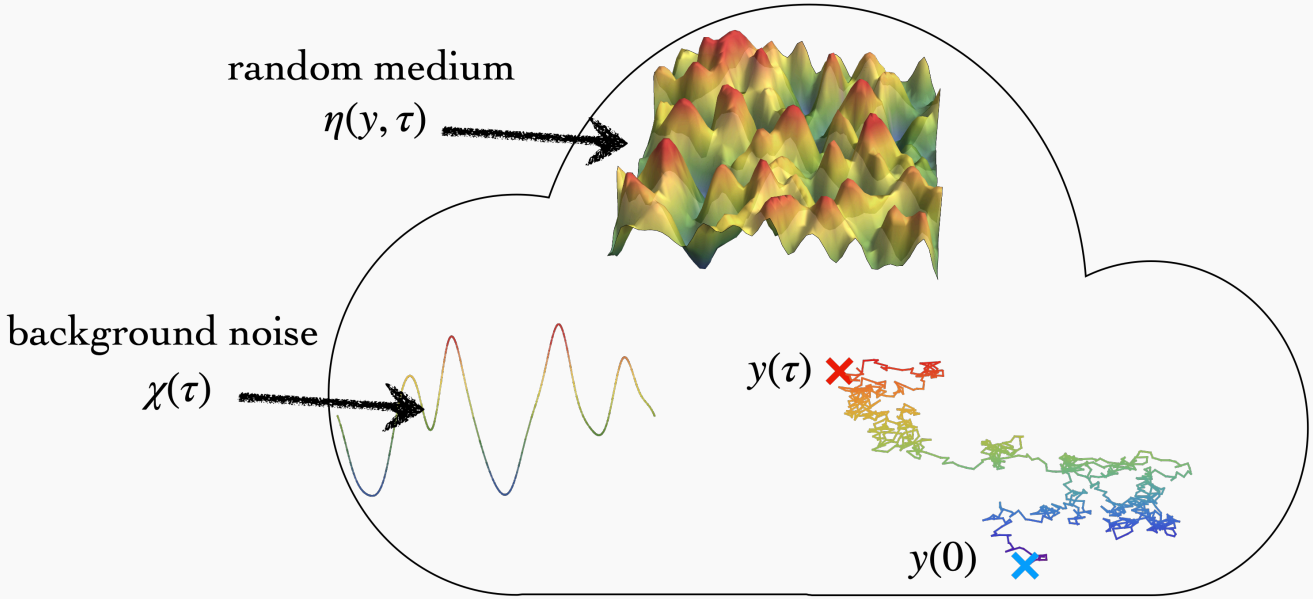
# Random walk of a particle in a time-dependent environment

Consider the diffusion of a particle (a tracer) at a position $y(\tau)$ convected by a centered Gaussian random field $\eta(y, \tau)$ which is white noise in space and time and by $\chi(\tau)$ which is a standard white noise in time.




The diffusion of the tracer is naturally described by a Langevin equation

$$\frac{\mathrm{d}y(\tau)}{\mathrm{d}t} = \sqrt{2}\eta(y(\tau), \tau) + \chi(\tau)$$

The noise $\chi(\tau)$ is also interpreted as a common random environment if there are multiple tracers: this will induce non-trivial correlations.

# Random walk of a particle in a time-dependent environment

For a given realisation realisation of the noise $\eta$, we can study the probability density function of finding the particle at a given position

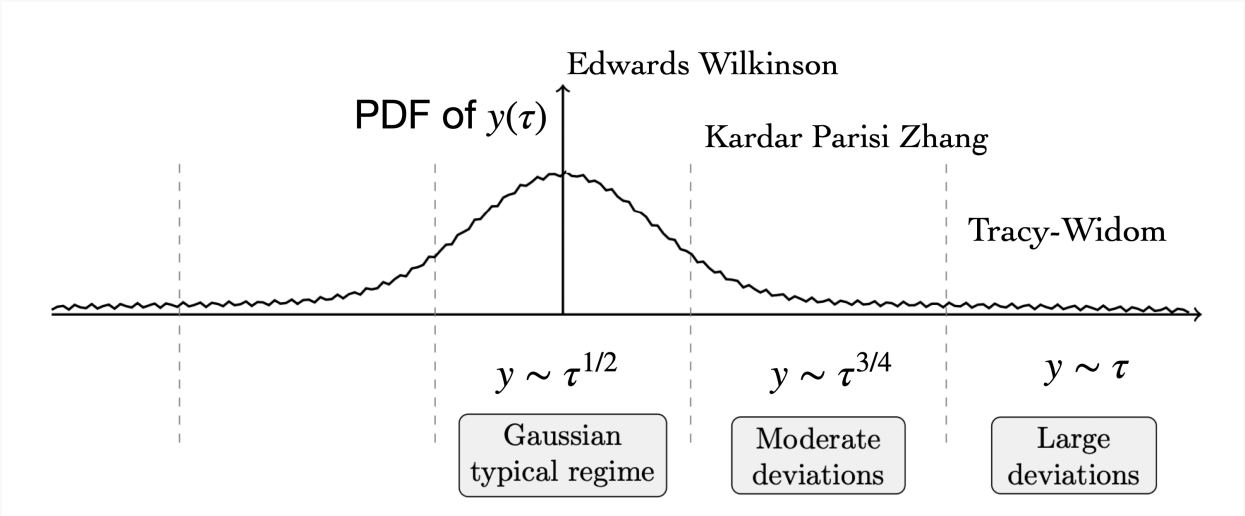
$$q_\eta(y,\tau) = \langle \delta(y(\tau) - y) \rangle_\chi$$

where $\langle \cdot \rangle_\chi$ denotes an averaging with respect to the background noise. The Fokker-Planck equation associated to the original diffusion yields

$$\partial_\tau q_\eta(y,\tau) = \partial_y^2 q_\eta(y,\tau) - \partial_y(\sqrt{2}\eta(y,\tau) q_\eta(y,\tau)) \,.$$

The average behavior of the density is rather dull and $\overline{q}_\eta$ yields the standard diffusion so that the standard deviation of the tracer's displacement is $y \sim \sqrt{\tau}$.

What about the sample-to-sample fluctuations of the density ?



## What happens for diffusions in atypical directions ?

Studying the system in an atypical direction

$$y = v\tau \quad \text{or,} \quad y \sim \tau^{\alpha}, \quad \alpha \in (1/2, 1]$$

is equivalent to inducing an asymmetry which changes the universality class of a system (e.g. SSEP to ASEP, Gaussian to KPZ...).

This problem is called the **extremal diffusion**. It allows to study diffusion beyond Einstein's Gaussian predictions and probe the cross-over between different universality classes.

**Some precedent works done by**

- Barraquand, Corwin (1503.04117)
- Thierry, Le Doussal (1705.05159)
- Barraquand, Rychnovsky (1905.10280)
- Barraquand, Le Doussal (1912.11085)
- Hass, Carroll-Godfrey, Corwin, Corwin (2205.02265)

# Diffusive systems and the Macroscopic Fluctuation Theory

The diffusion in time-dependent random medium is part of a large class of model that can be studied in the framework of the Macroscopic Fluctuation Theory. This theory describes the hydrodynamic limit of numerous discrete models:

- Simple exclusion process
- Zero range process
- Random average process
- Kipnis-Marchioro-Presutti
- Hard Brownian particles
- Hard rod gas
- Double exclusion process

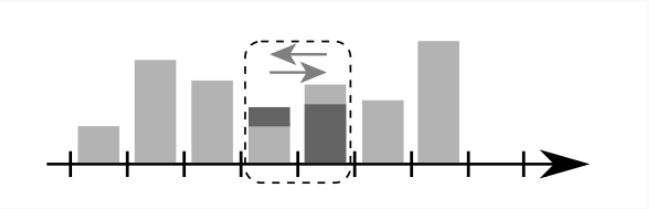


**Figure:** KMP model: neighbouring sites randomly redistribute energies at random times. taken from arXiv:2207.07549

The hydrodynamics are described by two transport coefficients (diffusion and mobility)

$$\partial_t q = \partial_y [D(q)\partial_y q - \sqrt{\sigma(q)}\eta(y,t)]$$

**Examples**

SSEP: $D = 1$, $\sigma(q) = 2q(1-q)$, KMP: $D = 1$, $\sigma(q) = 2q^2$.

## Diffusive systems and the Macroscopic Fluctuation Theory

Consider a particle starting from the origin $y(\tau = 0) = 0$ and study the statistics of the probability $Z(Y, T)$ that at time $\tau = T$ it is found to the right of $y = Y$

$$Z(Y, T) = \mathbb{P}(y(T) > Y | y(0) = 0) = \int_Y^{+\infty} \mathrm{d}y \, q_\eta(y, T)$$

with $q_\eta(y, 0) = \delta(y)$. The MFT allows to describe the large deviations of the density and current in the diffusive scaling

$$Y = \xi\sqrt{T}, \qquad T \gg 1, \quad \xi = \mathcal{O}(1)$$

where $\xi$ is the asymmetry parameter. The following **Large Deviation Principle** for the probability density of $Z(Y, T)$ is then predicted at large time $T \gg 1$

$$\mathcal{P}(Z) \sim \exp\big(-\sqrt{T}\hat{\Phi}(Z)\big)$$

**Our aim ? Calculate $\hat{\Phi}(Z)$**

When $\xi$ grows, the limit of the MFT is attained and we enter the early regime of the KPZ fluctuations.

# Some history and context

**Why studying the large deviations ?**

The large deviations of a diffusion model correspond to an excess or deficit of particles at some location in space. Let's take the example of an anode in a battery where electrons are detached to induce a current in a one-dimensional flows. If the anode is sufficiently big, disparate regions will see roughly independent flow. The maximal flow of the entire battery will be determined by the one-point tail behaviour. Since the current determines the battery performance, large deviations will dictate failure rates and lifetime.

**Long history of works within the MFT**

- Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim
- Derrida, Gershenfeld, Bodineau
- Meerson, Krapivsky, Smith, Bettelheim
- Grabsch, Poncet, Rizkallah, Illien, Bénichou
- Mallick, Moriya, Sasamoto
- Spohn
- Krajenbrink, Le Doussal

## Some history and context

Solutions of the MFT have been found historically through a variety of methods

- Stochastic integrability,
- Quantum integrability,
- Perturbation theory,
- Recursion of hierarchical equations for correlation functions.

These methods are often convoluted or ad-hoc and provide either limited solutions or solutions for very particular instances. Hence, a standard, versatile and simple method is still lacking.

We have developed a method based on **classical integrability** for the large deviations of the KPZ equations which also applies to the diffusion in time-dependent random medium.

*Inverse scattering of the Zakharov-Shabat system solves the weak noise theory of the Kardar-Parisi-Zhang equation*
arXiv:2103.17215 & Phys. Rev. Lett. **127**, 064101.

# Outline

**The crossover from the Macroscopic Fluctuation Theory to the Kardar-Parisi-Zhang equation controls the large deviations beyond Einstein's diffusion**

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1. The Martin-Siggia-Rose approach to the Macroscopic Fluctuation Theory: the landmark of stochastic field theory

2. The direct and inverse scattering transforms: the landmark of classical integrability

3. Completion of the large-deviation program

4. Matching with the large deviations of KPZ

## Probability distribution at large times

To obtain $\hat{\Phi}(Z)$, one defines an intermediate **Large Deviation Principle** through the following generating function

$$\mathbb{E}_\eta\left[\exp\left(-z\sqrt{T}Z\right)\right] \sim \exp\left(-\sqrt{T}\Psi(z)\right)$$
$$= \int \mathrm{d}Z\, \mathcal{P}(Z) \exp\left(-z\sqrt{T}Z\right)$$
$$= \int \mathrm{d}Z\ \exp\left(-\sqrt{T}[zZ + \hat{\Phi}(Z)]\right)$$

The large deviation function $\Phi(H)$ can be determined by a Legendre transform.

***Result (Large deviation problem)***

*For* $T \gg 1$,

$$\Psi(z) = \min_{Z\in[0,1]}[\hat{\Phi}(Z) + zZ]$$

*The rate function* $\Psi(z)$ *controls all the cumulants of* $Z(Y,T)$.

# Martin-Siggia-Rose formalism I

Starting from the Fokker-Planck equation

$$\partial_\tau q_\eta(y,\tau) = \partial_y^2 q_\eta(y,\tau) - \partial_y(\sqrt{2}\eta(y,\tau) q_\eta(y,\tau))\,.$$

taking the diffusive scaling $y = x\sqrt{T}$, $\tau = tT$, and $\sqrt{T}\, q_\eta(y,\tau) = Q_{\tilde{\eta}}(x,t)$ so that

$$\partial_t Q_{\tilde{\eta}} = \partial_x^2 Q_{\tilde{\eta}}(x,t) - T^{-1/4}\partial_x(\sqrt{2}\tilde{\eta}(x,t) Q_{\tilde{\eta}}(x,t))$$

where $Z(Y,T) = \int_\xi^{+\infty} \mathrm{d}x\, Q_{\tilde{\eta}}(x,1)$. Now $t \in [0,1]$ and if $T \gg 1$, the new standard white noise has a small magnitude: it is a weak noise.

We now write the generating function

$$\mathbb{E}_{\tilde{\eta}}\left[\exp\left(-z\sqrt{T}Z\right)\right]$$

where the expectation is taken with respect to the white noise

$$\mathcal{D}\tilde{\eta}\exp\left(-\frac{1}{2}\iint_{xt}\mathrm{d}x\mathrm{d}t\,\tilde{\eta}(x,t)^2\right)$$

# Martin-Siggia-Rose formalism II

After the introduction of a response field $\tilde{P}\sqrt{T}$, one then integrates over the noise $\tilde{\eta}$ and

**Result (MSR large deviation action)**

*overall the MSR action at long time reads*

$$\mathbb{E}_{\tilde{\eta}}\left[\exp\left(-z\sqrt{T}Z\right)\right] = \iint \mathcal{D}\tilde{Q}\mathcal{D}\tilde{P}e^{-\sqrt{T}(S[\tilde{P},\tilde{Q}]+z\int_0^1 \mathrm{d}t\delta(t-1)\int_\xi^{+\infty}\mathrm{d}x\tilde{Q}(x,t))}$$

*with*

$$S[\tilde{P},\tilde{Q}] = \int_0^1 \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x[\tilde{P}(\partial_t - \partial_x^2)\tilde{Q} - \tilde{Q}^2(\partial_x\tilde{P})^2]$$

*This action is amenable to a saddle-point evaluation!*

## Hydrodynamic description

The optimal height and noise verify the non-linear hydrodynamic system

$$\partial_t \tilde{Q} = \partial_x^2 \tilde{Q} - 2\partial_x(\tilde{Q}^2(\partial_x \tilde{P}))$$
$$-\partial_t \tilde{P} = \partial_x^2 \tilde{P} + 2\tilde{Q}(\partial_x \tilde{P})^2$$

Define the derivative of the response field

$$\tilde{R}(x,t) = \partial_x \tilde{P}(x,t)$$

Then it is transformed into the $\{R, Q\}$ system

$$\partial_t \tilde{Q} = \partial_x^2 \tilde{Q} - 2\partial_x(\tilde{Q}^2 \tilde{R})$$
$$-\partial_t \tilde{R} = \partial_x^2 \tilde{R} + 2\partial_x(\tilde{Q}\tilde{R}^2)$$

with the boundary conditions

$$\tilde{Q}(x, t=0) = \delta(x) \quad , \quad \tilde{R}(x, t=1) = -z\delta(x-\xi)$$

**Main takeway I**

This system is the derivative nonlinear Schrodinger equation. It can be solved **explicitly** without approximation.

# Intermediate summary of the large deviation problem

We finally apply a Galilean boost

$$Q(x,t) = \tilde{Q}(x+\xi t, t) e^{\frac{1}{2}x\xi + \frac{\xi^2}{4}t} \quad , \quad R(x,t) = \tilde{R}(x+\xi t, t) e^{-\frac{1}{2}x\xi - \frac{\xi^2}{4}t} \quad ,$$

with the interpolating system

$$\partial_t Q = \partial_x^2 Q - 2\partial_x(Q^2 R) + \xi Q^2 R$$
$$-\partial_t R = \partial_x^2 R + 2\partial_x(QR^2) + \xi QR^2$$

with initial / boundary conditions

$$\underbrace{Q(x,0) = \delta(x)}_{\text{initial density}}, \quad \underbrace{R(x,1) = -z e^{-\frac{\xi^2}{4}} \delta(x)}_{\text{derivative response field}}, \quad \underbrace{Z = \int_\xi^{+\infty} \mathrm{d}x\, \tilde{Q}_\eta(x,1)}_{\text{value of the conditioned probability}} \quad .$$

**MFT large deviation function**

The large deviation function is

$$\Psi'(z) = \int_\xi^{+\infty} \mathrm{d}x\, \tilde{Q}(x,1) = \int_0^{+\infty} \mathrm{d}x\, Q(x,1) e^{-\frac{1}{2}x\xi - \frac{\xi^2}{4}}$$

And now what ? The system is non-linear, not very friendly...

# That is where our heroes come to our rescue

**An exact solution for a derivative nonlinear Schrödinger equation[a)]**

David J. Kaup

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Alan C. Newell

*Department of Mathematics, Clarkson College of Technology, Potsdam, New York 13676*
(Received 12 August 1977)

A Generalization of Inverse Scattering Method

M. Wadati*, K. Konno** and Y.H. Ichikawa***

IPPJ- 381 March 1979

Received March 30,1979

Kaup, Newell, Wadati, Konno and Ichikawa tell us how to linearize the problem.

# The interpolating system is classically integrable!

We have the existence of a Lax pair: define a 2-component vector $\vec{v} = (v_1, v_2)^\intercal$ depending on $(x, t, k)$ (space, time, Fourier) and the linear differential system

$$\partial_x \vec{v} = U_1 \vec{v}, \qquad U_1 = \begin{pmatrix} -\frac{\mathrm{i}k}{2} & -(\frac{\xi}{2} - \mathrm{i}k)R(x,t) \\ Q(x,t) & \frac{\mathrm{i}k}{2} \end{pmatrix}$$

and

$$\partial_t \vec{v} = U_2 \vec{v}$$

***Result (Compatibility)***

*The compatibility equation is*

$$\partial_{xt}\vec{v} = \partial_{tx}\vec{v} \quad \textit{or} \quad \partial_t U_1 - \partial_x U_2 + [U_1, U_2] = 0.$$

*This is precisely the interpolating system.*

The spatial equation is an extension of the Zakharov-Shabat system with non-Hermitian potentials $R, Q$. The ZS system is the landmark of the AKNS class of integrable non-linear problems (comprising KdV, mKdV, NLS...) solvable using scattering theory.

# Definition of the scattering problem

Let $\vec{v} = e^{k^2 t/2}\phi$ with $\phi = (\phi_1, \phi_2)^\intercal$ and $\vec{v} = e^{-k^2 t/2}\bar{\phi}$ be two independent solutions of the linear problem such that

$$\phi \underset{x\to-\infty}{\simeq} \begin{pmatrix} e^{-\mathrm{i}kx/2} \\ 0 \end{pmatrix}, \qquad \bar{\phi} \underset{x\to-\infty}{\simeq} \begin{pmatrix} 0 \\ -e^{\mathrm{i}kx/2} \end{pmatrix}$$

and

$$\phi \underset{x\to+\infty}{\simeq} \begin{pmatrix} a(k,t)e^{-\frac{\mathrm{i}kx}{2}} \\ b(k,t)e^{\frac{\mathrm{i}kx}{2}} \end{pmatrix}, \quad \bar{\phi} \underset{x\to+\infty}{\simeq} \begin{pmatrix} \tilde{b}(k,t)e^{-\frac{\mathrm{i}kx}{2}} \\ -\tilde{a}(k,t)e^{\frac{\mathrm{i}kx}{2}} \end{pmatrix}$$

$(a, \tilde{a}, b, \tilde{b})$ are called the scattering amplitudes.

The following ratios define the reflection coefficients

$$r(k,t) = \frac{b(k,t)}{a(k,t)}, \qquad \tilde{r}(k,t) = \frac{\tilde{b}(k,t)}{\tilde{a}(k,t)}$$

# Scattering transforms

$$\{Q(x,t=0),R(x,t=1)\} \underset{\substack{\text{direct}\\ \text{scattering}\\ \text{transform}}}{\longrightarrow} \{a(k,t),b(k,t)\} \underset{\substack{\text{inverse}\\ \text{scattering}\\ \text{transform}}}{\longrightarrow} \{Q(x,t),R(x,t)\}$$

**Main takeway II**

Both of the direct and inverse scattering transforms are done **explicitly**.

Our plan:

1. Time dependence: $\{a(k,t),b(k,t)\}$
2. Fourier dependence: $\{a(k,t),b(k,t)\}$

# DST - Time-dependence of the scattering coefficients

Plugging

$$\phi \underset{x\to+\infty}{\simeq} \begin{pmatrix} a(k,t)e^{-\frac{ikx}{2}} \\ b(k,t)e^{\frac{ikx}{2}} \end{pmatrix} \ , \ \bar{\phi} \underset{x\to+\infty}{\simeq} \begin{pmatrix} \tilde{b}(k,t)e^{-\frac{ikx}{2}} \\ -\tilde{a}(k,t)e^{\frac{ikx}{2}} \end{pmatrix}$$

into the time equation

$$\partial_t \vec{v} = U_2 \vec{v}$$

One finds that

**Result (Time-dependence)**

$$\begin{cases} a(k,t) = a(k) \\ \tilde{a}(k,t) = \tilde{a}(k) \\ b(k,t) = b(k)e^{-k^2 t} \\ \tilde{b}(k,t) = \tilde{b}(k)e^{k^2 t} \end{cases}$$

*This is universal and independent of the potentials $R, Q$ !*

We also have the normalization $a\tilde{a} + b\tilde{b} = 1$, this is universal.

# DST - Fourier-dependence of the scattering coefficients

One then solves the spatial part at $t = 0$ and $t = 1$

$$\partial_x \vec{v} = U_1 \vec{v}, \qquad U_1 = \begin{pmatrix} -\frac{\mathrm{i}k}{2} & -(\frac{\xi}{2} - \mathrm{i}k)R(x,t) \\ Q(x,t) & \frac{\mathrm{i}k}{2} \end{pmatrix}$$

Using

$$Q(x,0) = \delta(x), \quad R(x,1) = -ze^{-\frac{\xi^2}{4}}\delta(x).$$

One finds that

**Result (Fourier-dependence)**

$$\begin{cases} b(k) = 1 \\ \tilde{b}(k) = -(\frac{\xi}{2} - \mathrm{i}k)ze^{-k^2 - \frac{\xi^2}{4}} \\ \tilde{a}(k) = 1 + (\frac{\xi}{2} - \mathrm{i}k)ze^{-\frac{\xi^2}{4}} Q_+(k) \\ a(k) = 1 + (\frac{\xi}{2} - \mathrm{i}k)ze^{-\frac{\xi^2}{4}} Q_-(k) \end{cases}$$

*where we have defined the half-Fourier transforms*

$$Q_\pm(k) = \int_{\mathbb{R}^\pm} \mathrm{d}x\, Q(x,1)e^{-\mathrm{i}kx}$$

## Some intermediate results

From the normalization relation one obtains

$$a(k)\tilde{a}(k) = 1 - b(k)\tilde{b}(k) = 1 + (\frac{\xi}{2} - \mathrm{i}k)ze^{-\frac{\xi^2}{4}}e^{-k^2}$$

which is solvable using scalar Riemann-Hilbert method in the complex plane.

We can deduce that the optimal density at final time $Q(x,1)$ has a discontinuity around 0 due to shocks

$$(1 - ze^{-\frac{\xi^2}{4}}Q(0^+,1))(1 + ze^{-\frac{\xi^2}{4}}Q(0^-,1)) = 1$$

The large deviation function can be read from the scattering amplitudes

$$\Psi'(z) = Q_+(k = -\mathrm{i}\frac{\xi}{2})e^{-\frac{\xi^2}{4}} = 1 - Q_-(k = -\mathrm{i}\frac{\xi}{2})e^{-\frac{\xi^2}{4}}$$

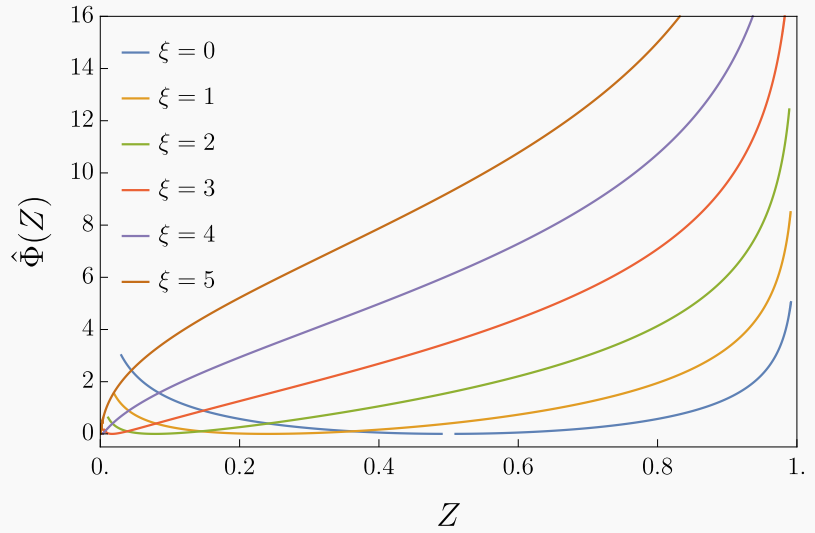
# What about the large deviation function ?

$$\mathbb{E}_{\tilde{\eta}}\left[\exp\left(-z\sqrt{T}Z\right)\right] \sim \exp\left(-\sqrt{T}\Psi(z)\right)$$

with

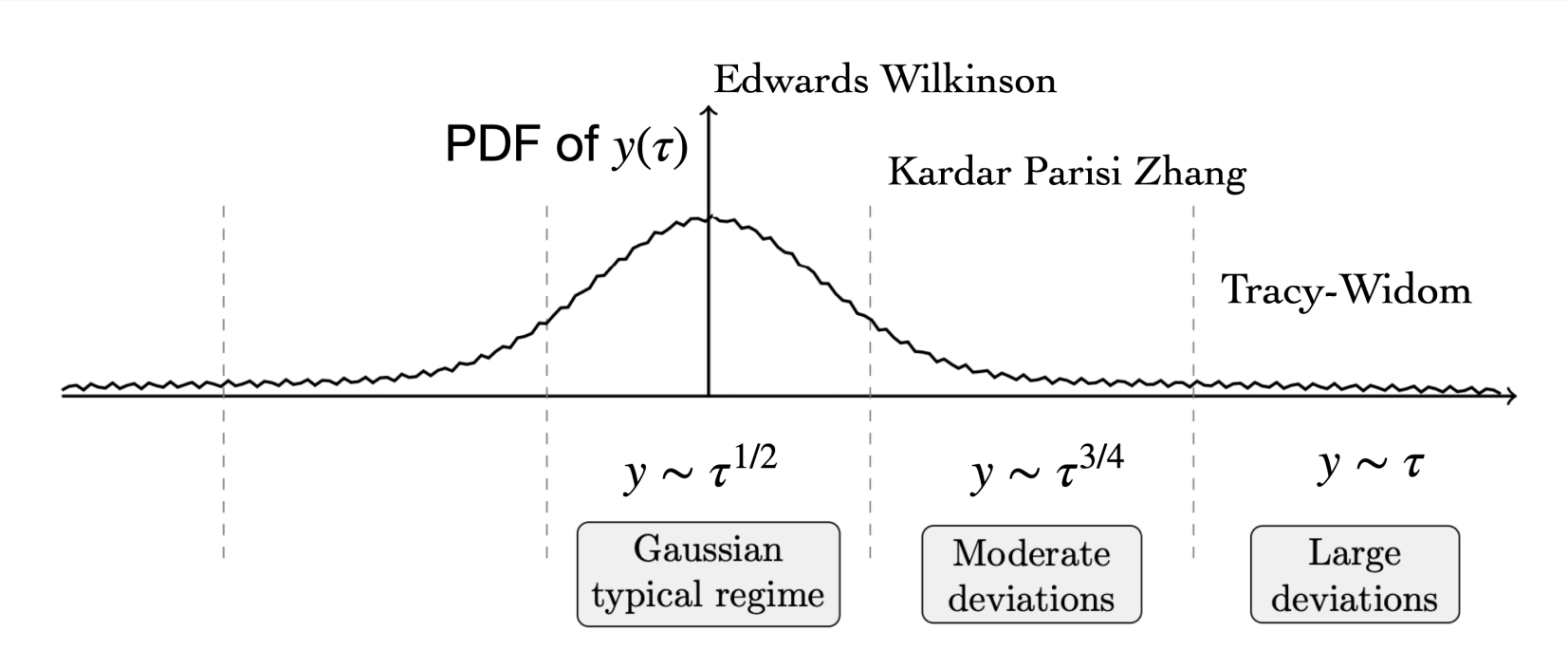
$$\Psi(z) = -\int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} \frac{\mathrm{Li}_2(z(\mathrm{i}q - \frac{\xi}{2})e^{-q^2 - \frac{\xi^2}{4}})}{(\mathrm{i}q - \frac{\xi}{2})^2} + z\Theta(-\xi)$$

where $\Theta$ is the Heaviside function with convention $\Theta(0) = 1/2$ and $\mathrm{Li}_2$ is the dilogarithm defined on some *adequate* Riemann sheet.




For $\xi$ large, we observe that $\hat{\Phi}(Z)$ is not convex but has a concave part!

# Leaving the MFT regime to approach the KPZ one




For $\xi \gg 1$, we should approach the KPZ regime. The mechanism is the last remaining piece of the puzzle.

# Leaving the MFT regime to approach the KPZ one

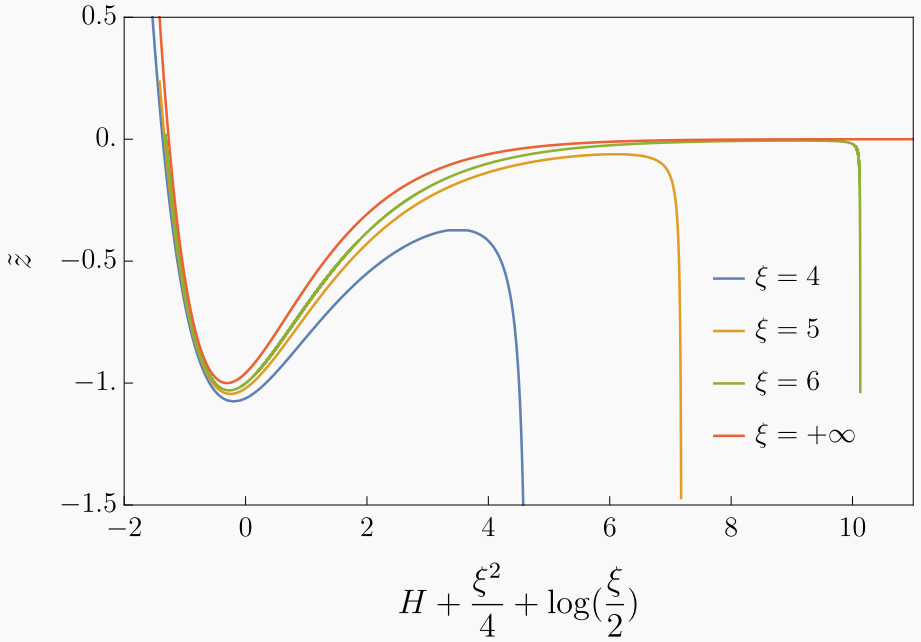
We now consider the limit where the tracer particle is located extremely far, i.e. $\xi \to +\infty$ and define the rescaled coupling $\tilde{z} = z\frac{\xi}{2}e^{-\frac{\xi^2}{4}}$.

$$\Psi(z) \simeq -\int_{\mathbb{R}} \frac{\mathrm{d}q}{2\pi} \frac{\mathrm{Li}_2(-z\frac{\xi}{2}e^{-q^2-\frac{\xi^2}{4}})}{(\frac{\xi}{2})^2} = \frac{4}{\xi^2}\Psi_{\mathrm{KPZ}}(\tilde{z})$$

where

$$\Psi_{\mathrm{KPZ}}(\tilde{z}) = -\frac{1}{\sqrt{4\pi}}\mathrm{Li}_{5/2}(-\tilde{z})$$

Recall that at optimality $\Psi'(z) = Z$, defining $H = \log Z$, we have



# Leaving the MFT regime to approach the KPZ one

Consider $H_{\mathrm{KPZ}} = h_{\mathrm{KPZ}}(0, T_{\mathrm{KPZ}}) + \log(\sqrt{T_{\mathrm{KPZ}}})$ the solution to the KPZ equation

$$\partial_t h_{\mathrm{KPZ}} = \partial_x^2 h_{\mathrm{KPZ}} + (\partial_x h_{\mathrm{KPZ}})^2 + \sqrt{2}\eta$$

with droplet initial condition evaluated at time $T_{\mathrm{KPZ}}$. It was proven for KPZ in 2015 that for $T_{\mathrm{KPZ}} \ll 1$ there is the following **Large Deviation Principle**

$$\overline{\exp(-\frac{\tilde{z}}{\sqrt{T_{\mathrm{KPZ}}}} e^{H_{\mathrm{KPZ}}})} = \exp\left(-\frac{\Psi_{\mathrm{KPZ}}(\tilde{z})}{\sqrt{T_{\mathrm{KPZ}}}}\right)$$

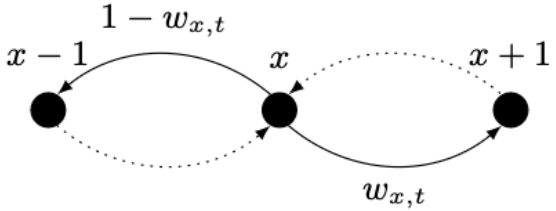
This result matches exactly with the MFT result using the following convergence for $\xi \gg 1$

$$T_{\mathrm{KPZ}} = \frac{Y^4}{16T^3} = \frac{\xi^4}{16T}, \qquad Z(Y,T) \simeq \frac{Y}{2T} e^{-\frac{Y^2}{4T}} e^{h_{\mathrm{KPZ}}(0, T_{\mathrm{KPZ}})}$$

The two results match perfectly, showing that no intermediate regime exists between the diffusive scaling $Y \sim \sqrt{T}$ and the finite-time KPZ equation scaling $Y \sim T^{3/4}$

# Extension to discrete models

It is natural to investigate whether the MFT associated to the RWRE can be related to a discrete random medium. We consider a discrete random walk in a Beta distributed environment known to converge to the continuum RWRE




$$P(w) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1-w)^{\beta-1}$$

Simulations using importance Monte Carlo sampling can be performed on this model.

**Probing the large deviations for the Beta random walk in random medium**


Alexander K. Hartmann,[1] Alexandre Krajenbrink,[2] and Pierre Le Doussal[3]

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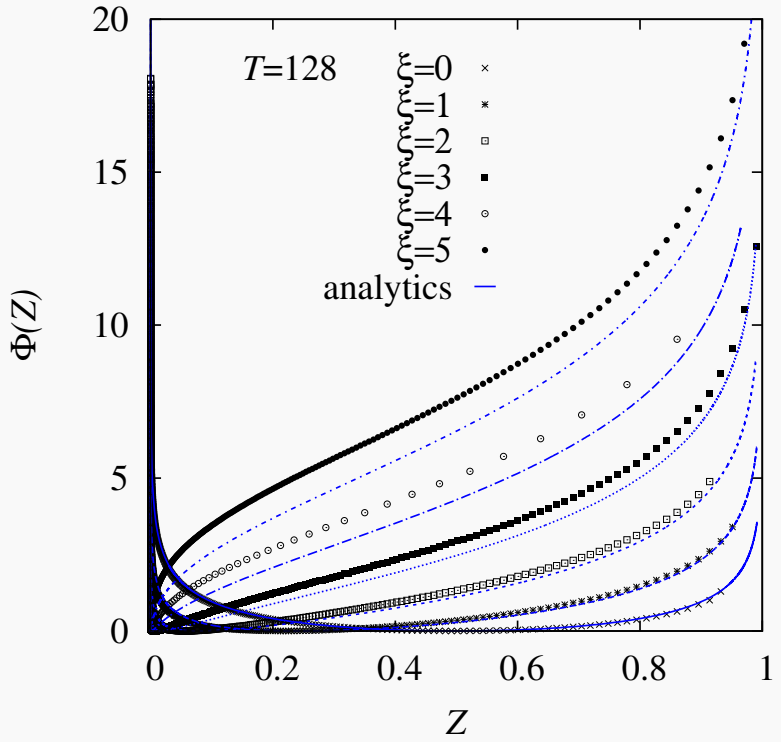
# Extension to discrete models

For a walk length $T = 128$ and various values of the asymmetry parameter $\xi$,

$$Z = \mathbb{P}\left(\frac{X(T)}{\sqrt{T/2}} > \xi\right)$$




It is argued that, up to a simple rescaling, this rate function of the discrete model is identical to the one obtained exactly for the continuum version of the model for large $T$.

# Extension to more general MFT

The framework introduced here is valid beyond the random walk in a time-dependent environment or KMP model.

- There exists a derivative Cole-Hopf transform which brings all quadratic MFT to the derivative NLS $\{R, Q\}$ system.
- Quadratic MFT are dual to other MFT models using the mapping [Rizkallah, Grabsch, Illien, Benichou, 2023]

$$\tilde{D}(q) = \frac{D(1/q)}{q^2}, \quad \tilde{\sigma}(q) = q\sigma(1/q)$$

- There exists a classical Jordan-Wigner gauge transformation [Wadati, Sogo, 1982] which brings the derivative NLS $\{R, Q\}$ system to the NLS $\{P, Q\}$ system which appears in the Weak Noise Theory of KPZ: this is applicable to all above MFT.

The quenched initial conditions are more easily solved using DNLS while the annealed ones are solved using the NLS system.

# Direct outlooks and other extensions

Exact solutions for the MFT have been obtained mainly by exploiting the (quantum/stochastic) integrability of the dynamics. We have derived here a nontrivial quantity directly from the underlying field theory, by exploiting the classical integrability of the saddle point equations. This provides a new tool on the Swiss knife to tackle the Macroscopic Fluctuation Theory. This is also the first work on the fine crossover between the MFT and KPZ.

- Extension to multiple tracers ✓
- Extension to more general initial conditions and models in the Macroscopic Fluctuation Theory *history in the making! work in progress...*
- Exact solution for the DNLS system using Fredholm determinants *history in the making! work in progress...*
- Are the KPZ fluctuations ubiquitous in the MFT with an asymmetry ?
- What if we quantize the Lax pairs ? *history in the making! work in progress...*

Thank you very much for listening!
Any questions ?