The crossover from the Macroscopic Fluctuation Theory to the Kardar-Parisi-Zhang equation controls the large deviations of diffusive systems

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Random walk of a particle in a time-dependent environment

Consider the diffusion of a particle (a tracer) at a position $y(\tau)$ convected by a centered Gaussian random field $\eta(y, \tau)$ which is white noise in space and time and by $\chi(\tau)$ which is a standard white noise in time.

The diffusion of the tracer is naturally described by a Langevin equation

$$\frac{dy(\tau)}{dt} = \sqrt{2} \eta(y(\tau), \tau) + \chi(\tau)$$

The noise $\chi(\tau)$ is also interpreted as a common random environment if there are multiple tracers: this will induce non-trivial correlations.
Random walk of a particle in a time-dependent environment

For a given realisation realisation of the noise $\eta$, we can study the probability density function of finding the particle at a given position

$$q_\eta(y, \tau) = \langle \delta(y(\tau) - y) \rangle_\chi$$

where $\langle \cdot \rangle_\chi$ denotes an averaging with respect to the background noise. The Fokker-Planck equation associated to the original diffusion yields

$$\partial_\tau q_\eta(y, \tau) = \partial_y^2 q_\eta(y, \tau) - \partial_y (\sqrt{2}\eta(y, \tau) q_\eta(y, \tau)).$$

The average behavior of the density is rather dull and $\bar{q}_\eta$ yields the standard diffusion so that the standard deviation of the tracer's displacement is $y \sim \sqrt{\tau}$.

What about the sample-to-sample fluctuations of the density?
What happens for diffusions in atypical directions?

Studying the system in an atypical direction

\[ y = v \tau \quad \text{or} \quad y \sim \tau^\alpha, \quad \alpha \in (1/2, 1] \]

is equivalent to inducing an asymmetry which changes the universality class of a system (e.g. SSEP to ASEP, Gaussian to KPZ...).

This problem is called the **extremal diffusion**. It allows to study diffusion beyond Einstein’s Gaussian predictions and probe the cross-over between different universality classes.

Some precedent works done by

- Barraquand, Corwin (1503.04117)
- Thierry, Le Doussal (1705.05159)
- Barraquand, Rychnovskyy (1905.10280)
- Barraquand, Le Doussal (1912.11085)
- Hass, Carroll-Godfrey, Corwin, Corwin (2205.02265)
Diffusive systems and the Macroscopic Fluctuation Theory

The diffusion in time-dependent random medium is part of a large class of model that can be studied in the framework of the Macroscopic Fluctuation Theory. This theory describes the hydrodynamic limit of numerous discrete models:

- Simple exclusion process
- Zero range process
- Random average process
- Kipnis-Marchioro-Presutti
- Hard Brownian particles
- Hard rod gas
- Double exclusion process

The hydrodynamics are described by two transport coefficients (diffusion and mobility)

\[ \partial_t q = \partial_y [D(q)\partial_y q - \sqrt{\sigma(q)}\eta(y, t)] \]

**Examples**

SSEP: \( D = 1, \sigma(q) = 2q(1 - q) \), KMP: \( D = 1, \sigma(q) = 2q^2 \).
Consider a particle starting from the origin \( y(\tau = 0) = 0 \) and study the statistics of the probability \( Z(Y, T) \) that at time \( \tau = T \) it is found to the right of \( y = Y \)

\[
Z(Y, T) = \mathbb{P}(y(T) > Y|y(0) = 0) = \int_{Y}^{+\infty} dy \, q_\eta(y, T)
\]

with \( q_\eta(y, 0) = \delta(y) \). The MFT allows to describe the large deviations of the density and current in the diffusive scaling

\[
Y = \xi \sqrt{T}, \quad T \gg 1, \quad \xi = \mathcal{O}(1)
\]

where \( \xi \) is the asymmetry parameter. The following Large Deviation Principle for the probability density of \( Z(Y, T) \) is then predicted at large time \( T \gg 1 \)

\[
\mathcal{P}(Z) \sim \exp(-\sqrt{T} \hat{\Phi}(Z))
\]

Our aim ? Calculate \( \hat{\Phi}(Z) \)

When \( \xi \) grows, the limit of the MFT is attained and we enter the early regime of the KPZ fluctuations.
Some history and context

Why studying the large deviations?

The large deviations of a diffusion model correspond to an excess or deficit of particles at some location in space. Let’s take the example of an anode in a battery where electrons are detached to induce a current in a one-dimensional flows. If the anode is sufficiently big, disparate regions will see roughly independent flow. The maximal flow of the entire battery will be determined by the one-point tail behaviour. Since the current determines the battery performance, large deviations will dictate failure rates and lifetime.

Long history of works within the MFT

- Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim
- Derrida, Gershenfeld, Bodineau
- Meerson, Krapivsky, Smith, Bettelheim
- Grabsch, Poncet, Rizkallah, Illien, Bénichou
- Mallick, Moriya, Sasamoto
- Spohn
- Krajenbrink, Le Doussal
Some history and context

Solutions of the MFT have been found historically through a variety of methods

- Stochastic integrability,
- Quantum integrability,
- Perturbation theory,
- Recursion of hierarchical equations for correlation functions.

These methods are often convoluted or ad-hoc and provide either limited solutions or solutions for very particular instances. Hence, a standard, versatile and simple method is still lacking.

We have developed a method based on **classical integrability** for the large deviations of the KPZ equations which also applies to the diffusion in time-dependent random medium.

*Inverse scattering of the Zakharov-Shabat system solves the weak noise theory of the Kardar-Parisi-Zhang equation*

The crossover from the Macroscopic Fluctuation Theory to the Kardar-Parisi-Zhang equation controls the large deviations beyond Einstein’s diffusion

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1. The Martin-Siggia-Rose approach to the Macroscopic Fluctuation Theory: the landmark of stochastic field theory

2. The direct and inverse scattering transforms: the landmark of classical integrability

3. Completion of the large-deviation program

4. Matching with the large deviations of KPZ
Probability distribution at large times

To obtain $\hat{\Phi}(Z)$, one defines an intermediate Large Deviation Principle through the following generating function

$$\mathbb{E}_{\eta} \left[ \exp \left( -z \sqrt{T} Z \right) \right] \sim \exp \left( -\sqrt{T} \Psi(z) \right)$$

$$= \int dZ \, \mathcal{P}(Z) \exp \left( -z \sqrt{T} Z \right)$$

$$= \int dZ \, \exp \left( -\sqrt{T} [zZ + \hat{\Phi}(Z)] \right)$$

The large deviation function $\Phi(H)$ can be determined by a Legendre transform.

**Result (Large deviation problem)**

For $T \gg 1$,

$$\Psi(z) = \min_{Z \in [0,1]} [\hat{\Phi}(Z) + zZ]$$

The rate function $\Psi(z)$ controls all the cumulants of $Z(Y, T)$.
Starting from the Fokker-Planck equation

$$\partial_\tau q_\eta(y, \tau) = \partial_y^2 q_\eta(y, \tau) - \partial_y(\sqrt{2\eta(y, \tau)} q_\eta(y, \tau)).$$

taking the diffusive scaling $y = x\sqrt{T}$, $\tau = tT$, and $\sqrt{T} q_\eta(y, \tau) = Q_{\tilde{\eta}}(x, t)$ so that

$$\partial_t Q_{\tilde{\eta}} = \partial_x^2 Q_{\tilde{\eta}}(x, t) - T^{-1/4} \partial_x(\sqrt{2\tilde{\eta}}(x, t) Q_{\tilde{\eta}}(x, t))$$

where $Z(Y, T) = \int_{\xi}^{+\infty} dx Q_{\tilde{\eta}}(x, 1)$. Now $t \in [0, 1]$ and if $T \gg 1$, the new standard white noise has a small magnitude: it is a weak noise.

We now write the generating function

$$\mathbb{E}_{\tilde{\eta}} \left[ \exp \left( -z\sqrt{T} Z \right) \right]$$

where the expectation is taken with respect to the white noise

$$\mathcal{D}_{\tilde{\eta}} \exp \left( -\frac{1}{2} \int\int_{xt} dx dt \tilde{\eta}(x, t)^2 \right)$$
After the introduction of a response field $\tilde{P} \sqrt{T}$, one then integrates over the noise $\tilde{\eta}$ and

\[
\mathbb{E}_{\tilde{\eta}} \left[ \exp \left( -z \sqrt{T} Z \right) \right] = \int \int \mathcal{D} \tilde{Q} \mathcal{D} \tilde{P} e^{-\sqrt{T} (S[\tilde{P}, \tilde{Q}]+z \int_0^1 dt \delta(t-1) \int_{-\infty}^{+\infty} dx \tilde{Q}(x,t))}
\]

with

\[
S[\tilde{P}, \tilde{Q}] = \int_0^1 dt \int_{\mathbb{R}} dx [\tilde{P}(\partial_t - \partial_x^2) \tilde{Q} - \tilde{Q}^2 (\partial_x \tilde{P})^2]
\]

This action is amenable to a saddle-point evaluation!
Hydrodynamic description

The optimal height and noise verify the non-linear hydrodynamic system

\[ \partial_t \tilde{Q} = \partial_x^2 \tilde{Q} - 2\partial_x (\tilde{Q}^2 \partial_x \tilde{P}) \]
\[ -\partial_t \tilde{P} = \partial_x^2 \tilde{P} + 2\tilde{Q}(\partial_x \tilde{P})^2 \]

Define the derivative of the response field

\[ \tilde{R}(x, t) = \partial_x \tilde{P}(x, t) \]

Then it is transformed into the \( \{R, Q\} \) system

\[ \partial_t \tilde{Q} = \partial_x^2 \tilde{Q} - 2\partial_x (\tilde{Q}^2 \tilde{R}) \]
\[ -\partial_t \tilde{R} = \partial_x^2 \tilde{R} + 2\partial_x (\tilde{Q} \tilde{R}^2) \]

with the boundary conditions

\[ \tilde{Q}(x, t = 0) = \delta(x) \quad , \quad \tilde{R}(x, t = 1) = -z\delta(x - \xi) \]

Main takeaway I

This system is the derivative nonlinear Schrodinger equation. It can be solved explicitly without approximation.
Intermediate summary of the large deviation problem

We finally apply a Galilean boost

\[ Q(x, t) = \tilde{Q}(x + \xi t, t)e^{\frac{1}{2}x\xi + \frac{\xi^2}{4} t}, \quad R(x, t) = \tilde{R}(x + \xi t, t)e^{-\frac{1}{2}x\xi - \frac{\xi^2}{4} t}, \]

with the interpolating system

\[
\begin{align*}
\partial_t Q &= \partial^2_x Q - 2\partial_x (Q^2 R) + \xi Q^2 R \\
-\partial_t R &= \partial^2_x R + 2\partial_x (QR^2) + \xi QR^2
\end{align*}
\]

with initial / boundary conditions

\[
\begin{align*}
Q(x, 0) &= \delta(x), & R(x, 1) &= -ze^{-\frac{\xi^2}{4}} \delta(x), & Z &= \int_{\xi}^{+\infty} \text{d}x \, \tilde{Q}_\eta(x, 1)
\end{align*}
\]

MFT large deviation function

The large deviation function is

\[
\Psi'(z) = \int_{\xi}^{+\infty} \text{d}x \, \tilde{Q}(x, 1) = \int_{0}^{+\infty} \text{d}x \, Q(x, 1)e^{-\frac{1}{2}x\xi - \frac{\xi^2}{4}}
\]

And now what? The system is non-linear, not very friendly...
That is where our heroes come to our rescue

An exact solution for a derivative nonlinear Schrödinger equation

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A Generalization of Inverse Scattering Method

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Kaup, Newell, Wadati, Konno and Ichikawa tell us how to linearize the problem.
The interpolating system is classically integrable!

We have the existence of a Lax pair: define a 2-component vector $\vec{v} = (v_1, v_2)^T$ depending on $(x, t, k)$ (space, time, Fourier) and the linear differential system

$$\partial_x \vec{v} = U_1 \vec{v}, \quad U_1 = \begin{pmatrix} -\frac{i k}{2} & -\left(\frac{\xi}{2} - i k\right) R(x, t) \\ Q(x, t) & \frac{i k}{2} \end{pmatrix}$$

and

$$\partial_t \vec{v} = U_2 \vec{v}$$

**Result (Compatibility)**

The compatibility equation is

$$\partial_{xt} \vec{v} = \partial_{tx} \vec{v} \quad \text{or} \quad \partial_t U_1 - \partial_x U_2 + [U_1, U_2] = 0.$$

This is precisely the interpolating system.

The spatial equation is an extension of the Zakharov-Shabat system with non-Hermitian potentials $R, Q$. The ZS system is the landmark of the AKNS class of integrable non-linear problems (comprising KdV, mKdV, NLS...) solvable using scattering theory.
Definition of the scattering problem

Let $\vec{v} = e^{k^2t/2}\phi$ with $\phi = (\phi_1, \phi_2)^T$ and $\vec{v} = e^{-k^2t/2}\bar{\phi}$ be two independent solutions of the linear problem such that

$$\phi \xrightarrow{x \to -\infty} \begin{pmatrix} e^{-ik\chi/2} \\ 0 \end{pmatrix}, \quad \bar{\phi} \xrightarrow{x \to -\infty} \begin{pmatrix} 0 \\ -e^{ik\chi/2} \end{pmatrix}$$

and

$$\phi \xrightarrow{x \to +\infty} \begin{pmatrix} a(k, t)e^{-ik\chi/2} \\ b(k, t)e^{ik\chi/2} \end{pmatrix}, \quad \bar{\phi} \xrightarrow{x \to +\infty} \begin{pmatrix} \tilde{b}(k, t)e^{-ik\chi/2} \\ -\tilde{a}(k, t)e^{ik\chi/2} \end{pmatrix}$$

$(a, \tilde{a}, b, \tilde{b})$ are called the scattering amplitudes.

The following ratios define the reflection coefficients

$$r(k, t) = \frac{b(k, t)}{a(k, t)}, \quad \tilde{r}(k, t) = \frac{\tilde{b}(k, t)}{\tilde{a}(k, t)}$$
Scattering transforms

\{Q(x, t = 0), R(x, t = 1)\} \rightarrow \{a(k, t), b(k, t)\} \rightarrow \{Q(x, t), R(x, t)\}

Main takeaway II

Both of the direct and inverse scattering transforms are done explicitly.

Our plan:

1. Time dependence: \{a(k, t), b(k, t)\}
2. Fourier dependence: \{a(k, t), b(k, t)\}
Plugging

\[
\phi \quad \sim \quad \begin{pmatrix} a(k, t) e^{-\frac{ikx}{2}} \\ b(k, t) e^{\frac{ikx}{2}} \end{pmatrix}, \quad \bar{\phi} \quad \sim \quad \begin{pmatrix} \tilde{b}(k, t) e^{-\frac{ikx}{2}} \\ -\tilde{a}(k, t) e^{\frac{ikx}{2}} \end{pmatrix}
\]

into the time equation

\[
\partial_t \vec{v} = U_2 \vec{v}
\]

One finds that

\[
\begin{align*}
a(k, t) &= a(k) \\
\tilde{a}(k, t) &= \tilde{a}(k) \\
b(k, t) &= b(k) e^{-k^2 t} \\
\tilde{b}(k, t) &= \tilde{b}(k) e^{k^2 t}
\end{align*}
\]

This is universal and independent of the potentials \( R, Q \)!

We also have the normalization \( a\tilde{a} + b\tilde{b} = 1 \), this is universal.
DST - Fourier-dependence of the scattering coefficients

One then solves the spatial part at $t = 0$ and $t = 1$

\[ \partial_x \vec{v} = U_1 \vec{v}, \quad U_1 = \begin{pmatrix} -\frac{ik}{2} & -\left(\frac{\xi}{2} - ik\right)R(x, t) \\ Q(x, t) & \frac{ik}{2} \end{pmatrix} \]

Using

\[ Q(x, 0) = \delta(x), \quad R(x, 1) = -ze^{-\frac{\xi^2}{4}} \delta(x). \]

One finds that

**Result (Fourier-dependence)**

\[
\begin{cases}
    b(k) = 1 \\
    \tilde{b}(k) = -\left(\frac{\xi}{2} - ik\right)ze^{-k^2 - \frac{\xi^2}{4}} \\
    \tilde{a}(k) = 1 + \left(\frac{\xi}{2} - ik\right)ze^{-\frac{\xi^2}{4}} Q_+(k) \\
    a(k) = 1 + \left(\frac{\xi}{2} - ik\right)ze^{-\frac{\xi^2}{4}} Q_-(k)
\end{cases}
\]

where we have defined the half-Fourier transforms

\[ Q_\pm(k) = \int_{\mathbb{R}^\pm} dx \, Q(x, 1)e^{-ikx} \]
Some intermediate results

From the normalization relation one obtains

\[ a(k)\tilde{a}(k) = 1 - b(k)\tilde{b}(k) = 1 + (\frac{\xi}{2} - ik)ze^{-\frac{\xi^2}{4}}e^{-k^2} \]

which is solvable using scalar Riemann-Hilbert method in the complex plane.

We can deduce that the optimal density at final time \( Q(x, 1) \) has a discontinuity around 0 due to shocks

\[ (1 - ze^{-\frac{\xi^2}{4}}Q(0^+, 1))(1 + ze^{-\frac{\xi^2}{4}}Q(0^-, 1)) = 1 \]

The large deviation function can be read from the scattering amplitudes

\[ \Psi'(z) = Q_+(k = -i\frac{\xi}{2})e^{-\frac{\xi^2}{4}} = 1 - Q_-(k = -i\frac{\xi}{2})e^{-\frac{\xi^2}{4}} \]
What about the large deviation function?

\[ \mathbb{E}_{\tilde{\eta}} \left[ \exp \left( -z \sqrt{T} Z \right) \right] \sim \exp \left( -\sqrt{T} \Psi(z) \right) \]

with

\[ \Psi(z) = -\int_{\mathbb{R}} \frac{dq}{2\pi} \frac{\text{Li}_2 \left( z(iq - \frac{\xi}{2})e^{-q^2 - \frac{\xi^2}{4}} \right)}{(iq - \frac{\xi}{2})^2} + z\Theta(-\xi) \]

where \( \Theta \) is the Heaviside function with convention \( \Theta(0) = 1/2 \) and \( \text{Li}_2 \) is the dilogarithm defined on some adequate Riemann sheet.

For \( \xi \) large, we observe that \( \hat{\Phi}(Z) \) is not convex but has a concave part!
Leaving the MFT regime to approach the KPZ one

For $\xi \gg 1$, we should approach the KPZ regime. The mechanism is the last remaining piece of the puzzle.
Leaving the MFT regime to approach the KPZ one

We now consider the limit where the tracer particle is located extremely far, i.e. \( \xi \to +\infty \) and define the rescaled coupling \( \tilde{z} = z \frac{\xi}{2} e^{-\frac{\xi^2}{4}} \).

\[
\Psi(z) \simeq -\int_{\mathbb{R}} \frac{dq}{2\pi} \frac{\text{Li}_2(-z \frac{\xi}{2} e^{-q^2 - \frac{\xi^2}{4}})}{(\frac{\xi}{2})^2} = \frac{4}{\xi^2} \psi_{\text{KPZ}}(\tilde{z})
\]

where

\[
\psi_{\text{KPZ}}(\tilde{z}) = -\frac{1}{\sqrt{4\pi}} \text{Li}_{5/2}(-\tilde{z})
\]

Recall that at optimality \( \Psi'(z) = Z \), defining \( H = \log Z \), we have
Leaving the MFT regime to approach the KPZ one

Consider $H_{\text{KPZ}} = h_{\text{KPZ}}(0, T_{\text{KPZ}}) + \log(\sqrt{T_{\text{KPZ}}})$ the solution to the KPZ equation

$$\partial_t h_{\text{KPZ}} = \partial_x^2 h_{\text{KPZ}} + (\partial_x h_{\text{KPZ}})^2 + \sqrt{2}\eta$$

with droplet initial condition evaluated at time $T_{\text{KPZ}}$. It was proven for KPZ in 2015 that for $T_{\text{KPZ}} \ll 1$ there is the following Large Deviation Principle

$$\exp(-\frac{\tilde{z}}{\sqrt{T_{\text{KPZ}}}} e^{H_{\text{KPZ}}}) = \exp\left(-\frac{\Psi_{\text{KPZ}}(\tilde{z})}{\sqrt{T_{\text{KPZ}}}}\right)$$

This result matches exactly with the MFT result using the following convergence for $\xi \gg 1$

$$T_{\text{KPZ}} = \frac{Y^4}{16 T^3} = \frac{\xi^4}{16 T}, \quad Z(Y, T) \sim \frac{Y}{2T} e^{-\frac{Y^2}{4T}} e^{h_{\text{KPZ}}(0, T_{\text{KPZ}})}$$

The two results match perfectly, showing that no intermediate regime exists between the diffusive scaling $Y \sim \sqrt{T}$ and the finite-time KPZ equation scaling $Y \sim T^{3/4}$
Extension to discrete models

It is natural to investigate whether the MFT associated to the RWRE can be related to a discrete random medium. We consider a discrete random walk in a Beta distributed environment known to converge to the continuum RWRE

\[ P(w) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1 - w)^{\beta-1} \]

Simulations using importance Monte Carlo sampling can be performed on this model.

Probing the large deviations for the Beta random walk in random medium

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Extension to discrete models

For a walk length $T = 128$ and various values of the asymmetry parameter $\xi$, 

$$Z = \mathbb{P}\left( \frac{X(T)}{\sqrt{T/2}} > \xi \right)$$

It is argued that, up to a simple rescaling, this rate function of the discrete model is identical to the one obtained exactly for the continuum version of the model for large $T$. 
Extension to more general MFT

The framework introduced here is valid beyond the random walk in a time-dependent environment or KMP model.

- There exists a derivative Cole-Hopf transform which brings all quadratic MFT to the derivative NLS $\{R, Q\}$ system.

- Quadratic MFT are dual to other MFT models using the mapping [Rizkallah, Grabsch, Illien, Benichou, 2023]

$$\tilde{D}(q) = \frac{D(1/q)}{q^2}, \quad \tilde{\sigma}(q) = q\sigma(1/q)$$

- There exists a classical Jordan-Wigner gauge transformation [Wadati, Sogo, 1982] which brings the derivative NLS $\{R, Q\}$ system to the NLS $\{P, Q\}$ system which appears in the Weak Noise Theory of KPZ: this is applicable to all above MFT.

The quenched initial conditions are more easily solved using DNLS while the annealed ones are solved using the NLS system.
Direct outlooks and other extensions

Exact solutions for the MFT have been obtained mainly by exploiting the (quantum/stochastic) integrability of the dynamics. We have derived here a nontrivial quantity directly from the underlying field theory, by exploiting the classical integrability of the saddle point equations. This provides a new tool on the Swiss knife to tackle the Macroscopic Fluctuation Theory. This is also the first work on the fine crossover between the MFT and KPZ.

- Extension to multiple tracers ✔️
- Extension to more general initial conditions and models in the Macroscopic Fluctuation Theory history in the making! work in progress...
- Exact solution for the DNLS system using Fredholm determinants history in the making! work in progress...
- Are the KPZ fluctuations ubiquitous in the MFT with an asymmetry ?
- What if we quantize the Lax pairs ? history in the making! work in progress...

Thank you very much for listening!
Any questions ?